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DIFFRACTION OF A HYDROMAGNETIC WAVE BY A HALF PLANE

by

Clifford S. Gardner
September 30, 1960

AEC Research and Development Report

NEW YORK UNIVERSITY

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Magneto-Fluid Dynamics Division
Institute of Mathematical Sciences
New York University

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MATHEMATICS

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ABSTRACT

The diffraction of a weak hydromagnetic shock wave by a rigid perfectly conducting half plane is considered. The medium is assumed to be a perfect compressible fluid with perfect electrical conductivity, permeated by a uniform magnetic field. The medium occupies the space outside a rigid perfectly conducting half plane. The reflected and diffracted waves resulting after an incident weak shock wave strikes the half plane are determined, in the linear or acoustic approximation. The problem is solved by means of an extension of the method of conical fields which is well known in the linearized theory of supersonic aerodynamics. In this paper we treat the case where the undisturbed magnetic field is parallel to the half-plane obstacle.

The results show the development in time, toward the ultimate steady state, of the phenomena associated with the reflection of a wave from a full plane. In this way a peculiar feature of hydromagnetic waves is elucidated. Namely, that sometimes the reflected wave is on what appears to be the wrong side of the obstacle, the opposite side from the incident wave.

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DIFFRACTION OF A HYDROMAGNETIC WAVE BY A HALF PLANE

1. Introduction.

In this paper we solve for the diffracted wave which results when a weak hydromagnetic shock impinges on a rigid perfectly conducting half plane.

We assume a medium which is a compressible perfect fluid (no heat conduction, no viscosity) and a perfect electrical conductor. In the undisturbed state the fluid is at rest and is permeated by a constant magnetic field $\vec{B}_o$. The fluid extends through all space around an obstacle, in the shape of a half plane, made of a rigid perfectly conducting material. Then, possibly due to a distant disturbance, a weak plane discontinuous wave or weak shock propagates through the fluid. We are interested in the diffracted wave caused by the interference of the half plane obstacle after the incident wave arrives. We assume that the amplitude of the disturbance of the fluid is small, so that it is appropriate to employ the linearized equations of motion.

We consider only a special case of the diffraction problem. If cartesian coordinates are so chosen that the obstacle is the half plane $x > 0$, $y = 0$, then the case we consider can be defined by saying that $\vec{B}_o$ is in the direction of the x-axis, and the direction of propagation of the incident wave is in the x y-plane. Other cases can be treated by our method. This treatment, however, is considerably more complicated, particularly if $\vec{B}_o$ is not parallel to the obstacle.

Our method is based on the fact that considerations of symmetry and similarity show that the solution of the problem depends only on two cartesian coordinates x, y and the time t, and only by way of the ratios x/t, y/t. The field of flow is a conical field in (x,y,t) space. We introduce formulas which represent the solution in terms of a single analytic function of a complex variable z. Here the z-plane is obtained from the plane of (x/t, y/t) by a non-conformal mapping. Using this representation, we obtain a problem of potential theory in the z-plane. We solve the potential problem and thereby the original wave diffraction problem.

The method of conical fields has been used to solve a great variety of problems in the linearized theory of supersonic flow. This method was introduced by Busemann(1). We have adapted this method to the linear theory of hydromagnetic waves, in which a more complicated system of differential equations is involved. Our approach is more closely related to that of Goldstein and Ward(2) than to that of Busemann. The starting point is the representation of an arbitrary solution as a superposition of plane-wave solutions (including both complex and real plane waves). The conical-field method has been applied to diffraction problems in electromagnetic theory and acoustics by Keller and Blank(3). The present work is thus a partial generalization of the work of Keller and Blank. In devising the correct formulation of the conical-field method for hydromagnetic waves, the author was guided by the recent results of H. Weitzner(4). Weitzner has determined explicit formulas for the fundamental solution of the hydromagnetic equations in the two-dimensional case. Examination of the structure of Weitzner's formulas revealed the right way to generalize the conical-field method to hydromagnetic-wave problems.

The author wishes to thank Dr. H. Weitzner for many illuminating discussions.

2. Equations of Motion.

The state of motion of the kind of fluid we consider is specified by

the pressure p and density ρ
the fluid velocity $\vec{u}$
the magnetic vector $\vec{B}$
the electric vector $\vec{E}$
the electric current density $\vec{J}$

These quantities are governed by the Lundquist equations(5), as follows:

Equation of continuity:

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot (\rho \vec{u}) = 0 \tag{1}$$

Ponderomotive equations:

$$\frac{\partial \vec{u}}{\partial t} + (\vec{u} \cdot \vec{\nabla}) \vec{u} + \frac{1}{\rho} \vec{\nabla} p + \frac{1}{\rho} \vec{B} \times \vec{J} = 0 \tag{2}$$

Maxwell's equations: (neglecting displacement current)

$$\vec{\nabla} \cdot \overline{B} = 0 \tag{3}$$

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times \vec{E} = 0 \tag{4}$$

$$\vec{\nabla}\times\vec{B} = \mu_o\vec{J} \qquad (5)$$

Ohm's Law for a perfect conductor:

$$\vec{E} + \vec{u}\times\vec{B} = 0 \qquad (6)$$

Adiabatic equation of state for the fluid:

$$p = p(\rho) \qquad (7)$$

We are concerned with small disturbances away from a state of rest; so it is appropriate to linearize the Lundquist equations by setting

$$\rho = \rho_o(1+\rho_1) \;;\; \vec{u} = \vec{u}_1 \;;\; \vec{B} = B_o(\vec{e}+\vec{B}_1) \qquad (8)$$

where ρ_o, $B_o\vec{e} = \vec{B}_o$ are the undisturbed values of density and magnetic field, and $\vec{e}$ is a unit vector. We insert (8) into the Lundquist equations and neglect terms of second degree in ρ_1, $\vec{u}_1$, $\vec{B}_1$ and their derivatives. Using (5), (6), (7) to eliminate $\vec{J}$, $\vec{E}$, p, we obtain the following: (to avoid clutter the subscripts are omitted).

$$\frac{\partial \rho}{\partial t} + \vec{\nabla} \cdot \vec{u} = 0 \tag{9}$$

$$\frac{\partial \vec{u}}{\partial t} + a^2 \vec{\nabla} \rho + A^2 \vec{e} \times (\vec{\nabla} \times \vec{B}) = 0 \tag{10}$$

$$\frac{\partial \vec{B}}{\partial t} + \vec{\nabla} \times (\vec{e} \times \vec{u}) = 0 \tag{11}$$

$$\vec{\nabla} \cdot \vec{B} = 0 \tag{12}$$

Here a, A are the undisturbed values of the sound speed and Alfvèn speed, respectively, defined by

$$a^2 = \frac{dp}{d\rho}, \quad A^2 = \frac{B_o^{\;2}}{\mu_o \rho_o} \tag{13}$$

3. Plane Waves.

A plane wave is a solution of (9), (10), (11), (12) in which $\rho, \vec{u}, \vec{B}$ depend only on the single parameter $\sigma = Vt - \vec{k} \cdot \vec{x}$. Here V is the propagation speed, and $\vec{k}$ is the unit vector in the direction of propagation. The properties of the various types of plane waves have been discussed by several authors, such as Grad(6), and Bazer and Fleischman(7). We shall review here, in a form adapted to our purposes, some of the pertinent results.

To study the plane-wave solutions, one inserts $\rho = \rho(\sigma)$, $\vec{u} = \vec{u}(\sigma)$, $\vec{B} = \vec{B}(\sigma)$ into (9), (10), (11), (12). One obtains a system of linear homogeneous equations for the derivatives $\rho', \vec{u}', \vec{B}'$, with coefficients depending on V and $\vec{k}$. As a compatibility condition one obtains a relation expressing how V depends on $\vec{k}$. One also obtains relations expressing how the ratios between ρ' and the various components of $\vec{u}'$ and $\vec{B}'$ depend on $\vec{k}$.

To state the results, it is convenient to introduce a cartesian coordinate system. Let the x-axis be in the direction of $\vec{B}_o = B_o\vec{e}$, and let the y-axis be in the plane defined by $\vec{B}_o$ and $\vec{k}$. Let θ denote the angle between $\vec{k}$ and $\vec{e}$, so that the x, y, and z components of $\vec{k}$ are $(\cos\theta, \sin\theta, 0)$. Let us denote the x, y, and z components of $\vec{B}$ by B_x, B_y, B_z; and the x, y, and z components of $\vec{u}$ by (u, v, w).

The dependence of V on θ is given by the equation

$$(V^2 - A^2\cos^2\theta)[V^2 - (A^2+a^2)V^2 + A^2a^2\cos^2\theta] = 0 \qquad (14)$$

In general (if $\cos^2\theta$ is not 0 or 1), this equation has three positive roots V_1, V_2, V_3, with $V_1 > V_2 > V_3$, corresponding to three types of plane wave which can propagate in the direction θ. The speeds V_1 and V_3 are the roots of the second factor of (14) and the corresponding types of plane wave are called the "fast wave" and the "slow wave". The root of the first factor is the speed of the "Alfvèn wave" or "intermediate wave".

The dependence of V on θ may be illustrated graphically by the diagram of Fig. 1. (This diagram is not drawn to scale, but is distorted for convenience.) In a given direction θ the fast wave travels with speed $V_1 = OP_1$ where P_1T_1 is normal to OP_1 and a tangent to the outer branch ABCD of the curve of Fig. 1, and OP_1 makes the angle θ with the x-axis or $\vec{B}_o$. The slow wave moves with speed $V_2 = OP_3$, where P_3T_3 is tangent to one of the three-cusped inner branches of the curve of Fig. 1, and normal to OP_3; and the Alfvèn wave travels with speed $V_2 = OP_2$, where AP_2 is normal to OP_2. The distance OA is the Alfvèn speed A, the distance Oa is the sound speed a, and the distance Oc is

$$c = \frac{Aa}{\sqrt{A^2+a^2}}$$

The diagram of Fig. 1 is drawn for the case A>a. For the case A<a the picture is similar, with A, a interchanged. For convenience we shall always assume A>a.

Consider the totality of plane waves which can propagate in all directions in the x y-plane, and suppose that the front of each such wave passes through 0 at time t = 0. Then at the time t = 1 the various wave fronts form three families of straight lines. The envelope of the fast-wave fronts is the curve ABCD. The envelope of the slow-wave fronts is made up of the two curves CFaE and GHI, together with the straight segment Gc. The envelope of the Alfvèn wave fronts is the straight segment CA.

The structure of the plane waves of the various types is given by the way that ρ and the components of u, B vary through the wave. It can be shown that for Alfvèn waves

$$d\rho = du = dv = dB_x = dB_y = 0 \qquad (15)$$

and

$$dw : dB_z = -V : \cos\theta \qquad (16)$$

For a fast or slow wave we have

$$dw = dB_z = 0 \qquad (17)$$

and

$$d\rho : du : dv : dB_x : dB_y = \frac{V(V^2-A^2)}{a^2 \sin\theta} : \frac{(V^2-a^2)\cos\theta}{V\sin\theta} : V : \sin\theta : -\cos\theta \quad (18)$$

In the limit where ρ,u,v, etc. vary abruptly and discontinuously from one set of values for $\sigma < 0$ to a different set of values for $\sigma > 0$ we have a plane discontinuous wave or weak shock. Then formulas like (16) or (18) relate the jumps in ρ,u,v, etc. across the wave front.

4. The Diffraction Problem.

We now describe the problem which will be solved in the present paper. We assume the following situation: a rigid, perfectly conducting half plane, which we take to be the plane $y = 0$, $x \geq 0$ is immersed in a space otherwise full of a perfectly conducting perfect fluid. Initially the fluid is at rest and permated by a constant uniform magnetic field in the direction of the x-axis. Then a plane wave, carrying perturbations of the density, velocity and magnetic field, possibly caused by some distant disturbance, propagates toward the half plane in a direction normal to the edge of the half plane. This situation is illustrated by Fig. 2. We suppose that the incident wave is a weak hydromagnetic shock wave. That is, we suppose that the functions $\rho(\sigma)$, $\vec{u}(\sigma)$, $\vec{B}(\sigma)$ are step functions, which change abruptly from zero for $\sigma < 0$ (ahead of the wave front) to constant final values for $\sigma > 0$ (behind the wave front). A full discussion of such weak discontinuous hydromagnetic waves has been given by Bazer and Fleischman (loc. cit.).

To solve our problem we must find a solution of the linearized Lundquist Equations (9)-(12) which satisfies the above-described initial conditions and satisfies the appropriate boundary conditions at the half plane. At a rigid perfectly conducting boundary, the normal component of the fluid velocity must be zero, and the tangential component of the electric field must be zero. Let $\vec{n}$ be a unit vector normal to such a boundary. Using Ohm's Law (Equation (6)) we see that for the linearized problem these boundary conditions are

$$\vec{n} \cdot \vec{u} = 0 \tag{19}$$

$$\vec{n} \times (\vec{e} \times \vec{u}) = (\vec{n} \cdot \vec{u})\vec{e} - (\vec{n} \cdot \vec{e})\vec{u} = 0 \tag{20}$$

If the undisturbed magnetic field is not parallel to the boundary, so that

$$\vec{n} \cdot \vec{e} \neq 0$$

then (19) and (20) are equivalent to

$$\vec{u} = 0 \qquad \text{at the boundary.}$$

But for the problem we are now considering, the undisturbed magnetic field is parallel to the boundary, so that

$$\vec{n} \cdot \vec{e} = 0$$

Thus for our problem the boundary conditions reduce to

$$\vec{n} \cdot \vec{u} = v = 0 \qquad \text{at } y = 0,\ x \geq 0 \qquad (21)$$

By (15), (16), we see that if the incident wave is an Alfvèn wave it will not be affected at all by the half plane. Thus we need consider only incident fast waves and slow waves. Then initially w and B_z are zero. Also, the other quantities characterizing the flow do not depend on the third cartesian coordinate z. So we assume that w and B_z are always zero and that ρ, u, v, B_x, B_y depend only on x, y, t which assumption is consistent with the various conditions of the problem. Furthermore, if we measure time from the instant at which the incident wave strikes the edge 0 of the half plane, then it is self-consistent to assume that ρ, u, v, B_x, B_y depend only on x/t, y/t; so that snapshots of the flow at successive instants would be one picture, being enlarged uniformly at a uniform rate.

We now indicate the situation for $t > 0$. Fig. 3 pertains to the case where the incident wave is a fast wave. The line MT_1P_1KN indicates the location of the incident wave front were it not for the interference of the half plane. Actually, as soon as the incident wave strikes the half plane, waves are sent out carrying the news. The line KL is the front of the wave which carries the news that the half plane is there. The news that the half plane has an end, at 0, is carried by all possible waves which can originate at 0 at time $t = 0$. So the front of the region where this information is disseminated is the curve AT_1BCDLA. Hence the picture shown in Fig. 3. In the region behind $MT_1BCDLKN$ we have the incident wave, as yet unaffected by the interference of the half plane.

In the region KAL we have a flow identical with that which would obtain if the half plane had no edge - i.e. the flow that would obtain if the half plane were a full plane. Thus in KAL the flow is like that after reflection of an incident fast wave by a plane. For similar reasons, there is no disturbance in the region KT_1A. Thus KL may be identified as the reflected wave front, and KT_1 as the shadow of the half plane. Inside AT_1BCDLA the fluid has received the intelligence that there is a half plane and that it has an edge at O; so in this region we have the diffracted wave. These remarks will be substantiated by exact calculations which will follow.

If the incident wave is a slow wave we have a different picture, shown in Fig. 4. The diffracted wave in this case outruns the incident wave MV and LN. Discontinuities in ρ, u, v, B_x, B_y occur across the segments VK, UK, T_3L; and there is no discontinuity across T_3K. Hence here UK should be called the reflected wave, and T_3K the shadow. The shadow is on the same side of the obstacle as the disturbance giving rise to the incident wave, and the reflected wave is on the opposite side of the obstacle!

5. Two-dimensional Equations of Motion.

To prepare for a precise mathematical formulation of our problem and for the calculations which will solve it, we procede with a more detailed study of the equations of motion. As we have seen, one may expect that for our problem we can assume that the components w, B_z of $\vec{u}, \vec{B}$ are zero, and that ρ, u, v, B_x, B_y depend only on x, y, t. Under these conditions, Equations (9)-(12) can be written as follows:

$$\frac{\partial \rho}{\partial t} + \frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0 \tag{22}$$

$$\frac{\partial u}{\partial t} + a^2 \frac{\partial \rho}{\partial x} = 0 \tag{23}$$

$$\frac{\partial v}{\partial t} + a^2 \frac{\partial \rho}{\partial y} + A^2 \left(\frac{\partial B_x}{\partial y} - \frac{\partial B_y}{\partial x}\right) = 0 \tag{24}$$

$$\frac{\partial B_x}{\partial t} + \frac{\partial v}{\partial y} = 0 \tag{25}$$

$$\frac{\partial B_y}{\partial t} - \frac{\partial v}{\partial x} = 0 \tag{26}$$

$$\frac{\partial B_x}{\partial x} + \frac{\partial B_y}{\partial y} = 0 \tag{27}$$

We note that solutions of this system can be represented in terms of a single function ψ, a sort of potential. By (27), we see that ψ exists such that

$$B_x = -\psi_{yyt} \tag{28}$$

$$B_y = \psi_{xyt} \tag{29}$$

(subscripts appended to ψ denote partial derivatives). Then from (25), (26), (24), (23) we see that these equations are satisfied if

$$v = -\psi_{ytt} \tag{30}$$

$$\rho = + \frac{1}{a^2} \frac{\partial}{\partial t} [\psi_{tt} - A^2(\psi_{xx} + \psi_{yy})] \tag{31}$$

$$u = - \frac{\partial}{\partial x} [\psi_{tt} - A^2(\psi_{xx} + \psi_{yy})] \tag{32}$$

Then from (22) we obtain the following equation for ψ:

$$\frac{\partial^4\psi}{\partial t^4} - (a^2 + A^2) \frac{\partial^2}{\partial t^2} \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) + a^2A^2 \frac{\partial^2}{\partial x^2} \left(\frac{\partial^2\psi}{\partial x^2} + \frac{\partial^2\psi}{\partial y^2}\right) = 0 \tag{33}$$

We must make a few remarks here about the propagation of discontinuities in a solution of the equations. This subject has been dealt with in detail by Bazer and Fleischman$^{(7)}$. Let us suppose we have a solution of (22)-(27) which tends to different limits as the point (x,y,t) tends to the two different sides of a surface S in (x,y,t) space, so that ρ,u,v,B_x,B_y have jump discontinuities across S. Of course, then the differential equations break down. One must use the laws of conservation of mass, momentum, and energy, and Ampere's law, which are integral relations. In this way it can be shown that at a discontinuity surface the jump in ρ,u,v,B_x,B_y, which we denote by $[\rho]$, $[u]$, $[v]$, $[B_x]$, $[B_y]$ must satisfy the following conditions:

$$n_t[\rho] + n_x[u] + n_y[v] = 0 \tag{34}$$

$$n_t[u] + a^2 n_x[\rho] = 0 \tag{35}$$

$$n_t[v] + a^2 n_y[\rho] + A^2(n_y[B_x] - n_x[B_y]) = 0 \tag{36}$$

$$n_t[B_x] - n_y[v] = 0 \tag{37}$$

$$n_t[B_y] + n_x[v] = 0 \tag{38}$$

$$n_x[B_x] + n_y[B_y] = 0 \tag{39}$$

there n_t, n_x, n_y are the components of a vector normal to S in (x,y,t) space. One elimination, it follows that

$$n_t^4 - (A^2 + a^2)n_t^2(n_x^2 + n_y^2) + A^2a^2n_x^2(n_x^2 + n_y^2) = 0 \qquad (40)$$

That is, S is a characteristic surface of the system (22)-(27) and of the single equation (33).

A characteristic surface can be shown to be the envelope of characteristic planes, which are fast or slow plane-wave fronts. A characteristic surface of special interest is the characteristic cone, which is the envelope of all characteristic planes through a point in (x,y,t) space. This the characteristic cone with vertex (0,0,0) is represented in the plane of (x/t,y/t) by the curves ABCD, ECF, GHI in Fig. 4.

6. Mathematical Formulation. Uniqueness Theorem.

We formulate the boundary problem for the reflected and diffracted waves; that is, for the total solution minus the incident wave. First we state regularity conditions required of ρ, u, v, B_x, B_y. We assume that these functions are differentiable in the open region R in (x,y,t) space gotten by subtracting certain characteristic surfaces and the half plane $x \geq 0$, $y = 0$ from all of (xyt) space. We assume that the functions ρ, u, etc. tend to continuous limits as (x,y,t) approaches either side of one of these surfaces, except at certain lines L_i. In Fig. 3 the discontinuity surfaces are: the characteristic cone ABCD, CFE, GHI, the plane wave fronts KT_1, KL, the half plane OK and in addition the plane segment OG — which is characteristic. The lines L_i are represented by K, T_1, L, G, and O, which last is the edge of the half plane. We assume that near the lines L_i, ρ, u, v, etc. are bounded — except near the line O. Near O we assume ρ, u, and v are bounded, but that B_x and B_y may become infinite in such a way that for any $t_o > 0$, $|B_x| < C(t_o)/\sqrt{r}$, $|B_y| < C(t_o)/\sqrt{r}$ for $t < t_o$, where $r = \sqrt{x^2+y^2}$.

We assume that in R the equations (22)-(27) are satisfied. We assume that at a characteristic surface of discontinuity the relations (34)-(39) are satisfied.

We assume that v satisfies on $x > 0$ $y = 0$ the following boundary condition:

$$\text{for } y = \pm 0, \quad 0 < x < V/\cos\theta, \quad v = 1$$

$$\text{for } y = \pm 0, \quad V/\cos\theta < x, \quad v = 0$$

Here of course, we assume that the incident wave travels with speed V in direction θ, and carries a value

$$v = -1.$$

We also assume, throughout that $0 < \theta < 90^\circ$. This is purely a matter of convenience.

Finally, we assume that

$$\text{for } t < 0,\ \rho = u = v = w = B_x = B_y = 0.$$

We also assume that for $t = t_0 > 0$ we have $\rho = u = v = w = B_x = B_y = 0$ outside of some bounded region of the xy plane.

This last assumption, which says merely that hydromagnetic waves travel with finite speed, can be proved to be a consequence of the other assumptions, but we shall not stop here to give the proof.

We now prove the uniqueness theorem; that the above problem can not have more than one solution. It is sufficient to show that the difference of two solutions is zero; and of course the difference of two solutions satisfies the homogeneous boundary condition

$$\text{for } y = \pm 0,\ x > 0,\ v = 0 \qquad (41)$$

The proof follows from conservation of energy. We multiply (22) by $a^2\rho$, (23) and (24) by u and v, (25) and (26) by A^2B_x and A^2B_y, and add the results. We obtain a relation which can be written as follows:

$$\frac{\partial}{\partial t}\left[\frac{1}{2}a^2\rho^2 + \frac{1}{2}(u^2 + v^2) + \frac{1}{2}A^2(B_x^2 + B_y^2)\right]$$

$$+ \frac{\partial}{\partial x}[a^2\rho u - A^2vB_y] + \frac{\partial}{\partial y}[v(a^2\rho + A^2B_x)] = 0 \qquad (42)$$

We now integrate this over all of (xyt) space with $t < t_o$. We obtain boundary terms at each surface S_i where ρ, u, v, B_x, B_y do not have continuous derivatives, or are discontinuous. These boundary terms arise from the second two terms of (42). The singularities at L_i contribute nothing, by virtue of the assumptions which were made about the behavior of ρ, u, v, B_x, B_y near L_i. The boundary term arising from the half plane $x > 0$, $y = 0$ is zero by virtue of the homogeneous boundary condition (41).

Also, the boundary term from a characteristic surface S is zero. For the integrand of the surface integral arising at S is

$$n_t[\tfrac{1}{2}\, a^2\bar{\rho}^2 + \tfrac{1}{2}(\bar{u}^2 + \bar{v}^2) + \tfrac{1}{2}\, A^2(\bar{B}_x^2 + \bar{B}_y^2)]$$

$$+ n_x[a^2\overline{\rho u} - A^2\overline{vB_y}] + n_y[\bar{v}(a^2\bar{\rho} + A^2\bar{B}_x)]$$

$$- n_t[\tfrac{1}{2}\, a^2\rho^2 + \tfrac{1}{2}\,(u^2 + v^2) + \tfrac{1}{2}\, A^2(B_x^2 + B_y^2)]$$

$$- n_x[a^2\rho u - A^2 vB_y] - n_y[v(a^2\rho + A^2B_x)] \qquad (43)$$

where (n_x, n_y, n_t) are the components of the unit normal to S in (x,y,t) space and $\bar{\rho}$, ρ; $\bar{u}$, u etc. are the values to which ρ, u etc. tend on the two sides of S. And, the expression (43) can be rewritten as follows:

$$\tfrac{1}{2}\, a^2(\rho + \bar{\rho})\left\{n_t[\rho] + n_x[u] + n_y[v]\right\}$$

$$+ \tfrac{1}{2}\,(u + \bar{u})\left\{n_t[u] + a^2\, n_x[\rho]\right\}$$

$$+ \tfrac{1}{2}\,(v + \bar{v})\left\{n_t[v] + a^2 n_y[\rho] + A^2 n_y[B_x] - A^2 m_x[B_y]\right\}$$

$$+ \tfrac{1}{2}\,(B_x + \bar{B}_x)\left\{n_t[B_x] + n_y[v]\right\}$$

$$+ \tfrac{1}{2}\,(B_y + \bar{B}_y)\left\{n_t[B_y] - n_x[v]\right\}$$

where $[\rho]$, $[u]$ are the jumps $\bar{\rho} - \rho$, $\bar{u} - u$ etc. in ρ, u, etc. Thus we see, by (34)-(39) that this boundary term is zero.

Hence after the integration has been carried out, we are left with

$$\iint_{t=t_o} [\tfrac{1}{2} a^2\rho^2 + \tfrac{1}{2}(u^2 + v^2) + \tfrac{1}{2} A^2(B_x^2 + B_y^2)]dxdy = 0 \qquad (44)$$

and (44) implies

$$\rho = u = v = B_x = B_y = 0$$

which proves the uniqueness theorem.

7. Representation of a Conical Field by Plane Waves.

We now develop the mathematical machinery by which our problem will be solved.

Let us consider a conical field; that is, a solution of (22)-(27) in which ρ, u, v, B_x, B_y depend only on x/t, y/t, or, put another way, in which ρ, u, v, B_x, B_y are homogeneous functions of x, y, t, of degree zero. We can construct such a conical field by superposing plane waves which are conical. Indeed, such a solution is given by

$$\psi(t,x,y) = -\frac{1}{6}\int_{z}^{\infty}[t - \lambda(s)x - \mu(s)y]^3 f(s)ds \tag{45}$$

where $\lambda(s)$, $\mu(s)$ are functions of s related by

$$F(\lambda,\mu) = 1 - (a^2 + A^2)(\lambda^2 + \mu^2) + a^2A^2\lambda^2(\lambda^2 + \mu^2) = 0 \tag{46}$$

and where z is a function of (x/t), (y/t) such that

$$t - \lambda(z)x - \mu(z)y = 0 \tag{47}$$

The function ψ defined by (45), (46), (47) satisfies Equation (33), as can be verified formally by direct differentiation, making use of the following result of differentiating (47):

$$\frac{\partial z}{\partial t} : \frac{\partial z}{\partial x} : \frac{\partial z}{\partial y} = 1 : -\lambda : -\mu$$

The solution ρ, u, v, B_x, B_y is given by third derivatives of ψ according to Equations (28)-(32). Since ψ evidently is homogeneous in x,y,t of degree 3, the third derivatives ρ, u, v, B_x, B_y are homogeneous of degree zero.

Equations (46) and (47) say that $1/\lambda$, $1/\mu$ are the intercepts with the x/t, y/t axes of a tangent through $P \equiv (x/t, y/t)$ to the characteristic cone of Fig. 1. In general through a given point $P \equiv (x/t, y/t)$ there are four such tangents. (Eliminating μ from (46), (47) one obtains a quartic equation for λ.) One can see from Fig. 1 that for P outside the outer branch of the characteristic cone the four tangents are real, being two fast waves and two slow waves. For P inside the inner branch, the four tangents are real, being four slow waves. For P between the inner and outer branch, one has only two real tangents - slow waves. The other tangents are complex. (This is, λ and μ are complex.)

Corresponding to the four different tangents, there are four possible formulas (45). We may surmise that by adding formulas of the four types one obtains the general conical field, expressed in terms of four functions f. This is true, although we shall not prove it here. To solve our problem it suffices to consider only the two complex waves associated with a point between the outer and inner branches of the characteristic cone. We may remark in passing that for a conical-field problem in which part of the boundary is not parallel to the undisturbed magnetic field, the waves corresponding to the real tangents also occur, and must be considered.

We now write down a formula for conical fields which represent a solution at points through which two tangents are complex. We define $\lambda(z)$ and $\mu(z)$ in the region in the z-plane outside the slits ca, AB, GK, CB (see Fig. 5)

by

$$\lambda = 1/z$$

$$\mu = \left(\frac{c}{aA}\right)\left(\frac{1}{z}\right)\sqrt{\frac{(z^2-A^2)(z^2-a^2)}{(z^2-c^2)}} \tag{48}$$

The square root occuring in μ is to be defined so as to be positive real for z real and on the upper part of the slit from $z = A$ to $z = +\infty$.

In (48), c is defined by

$$c^2 = \frac{a^2A^2}{a^2+A^2} \text{ .}$$

The functions (48) satisfy the relation (46), as can be verified directly.

The relation between (x/t, y/t) and z is then given by (47), or

$$\left(\frac{1}{z}\right)(x/t)+\left(\frac{c}{aA}\right)\left(\frac{1}{z}\right)\sqrt{\frac{(z^2-A^2)(z^2-a^2)}{(z^2-c^2)}}\,(y/t) = 1. \tag{49}$$

The real and imaginary parts of Equation (49) are two equations for (x/t), (y/t) in terms of z. This equation defines a differentiable one-to-one mapping of the open region R_1 between the outer branch ABCD and inner branches cFaE, GHKI of the characteristic cone of Fig. 1 into the slit z-plane of Fig. 5. (We shall prove this in the Appendix.) In this mapping the arc BAD in Fig. 1 is mapped into the slit BAD in Fig. 5 (where B,D are mapped into the point at infinity in the z-plane.) The arc BCD is mapped into the slit BCD in the z-plane. The curve cEaF is mapped into the slit cEaF, and the curve GHKI is mapped into the slit GHKI. A point M on the axis in R_1 (see Fig. 1) is mapped into a point M on the real line in Fig. 6. The origin O in Fig. 1 is mapped into the origin O in Fig. 5. Also the distance OM is not changed by the mapping. That is, the distance OM in Figure 1 is equal to the distance OM in Fig. 5. Also, near O the mapping of the (x/t), (y/t) plane into the z-plane approximates the identity mapping, in the sense that (if $z = \xi + i\eta$) we have

$$\begin{aligned} \xi &= (x/t)[1+O(|z|^4)] \\ \eta &= (y/t)[1+O(|z|^4)] \end{aligned} \qquad (50)$$

The intuitive reason for all this is that z has the following geometric interpretation: to determine z, draw through $P \equiv (x/t, y/t)$ one of the complex tangents to the characteristic cone. Then z is the intercept of that tangent on the x/t - axis. Since the point M is already on the x/t axis, any line through M intercepts the x/t axis at M. Hence OM in Fig. 5 is equal to OM in Fig. 1. As the point P approaches the outer branch ABCD of the characteristic cone at T_1, the two imaginary tangents become equal and therefore real. Hence T_1 is mapped on to the foot U_1 of the tangent T_1U_1. Similarly if P approaches T_3 on an inner branch aEcF of the characteristic cone, its image in the z-plane tends to a point U_3 on the real axis with OU_3 (Fig. 5) = OU_3 (Fig. 1) where T_3U_3 is tangent to aEcF. All these statments about the properties of the mapping will be proved in the Appendix.

Now we are prepared to state the formula with which we will solve the boundary problem. This formula is

$$\psi(x,y,t) = -\frac{1}{6}\,\mathrm{Im}\int_{z}^{\infty}[t - \lambda(s)x - \mu(s)y]^{3} f(s)\,ds \qquad (51)$$

where

$$\lambda(z) = \frac{1}{z}\,, \quad \mu(z) = \left(\frac{c}{aA}\right)\frac{1}{2}\sqrt{\frac{(z^2-A^2)(z^2-a^2)}{(z^2-c^2)}} \qquad (52)$$

and z is related to x/t, y/t by

$$\lambda(z)(x/t) + \mu(z)(y/t) = 1 \tag{53}$$

Here f(z) is required to be single-valued in the z-plane slit as follows: there are slits BC, KG on the negative real axis, from $-\infty$ to $-A$ and from $-a$ to $-c$. Also there is a slit from 0 to ∞ on the positive real axis. (The positive real axis is in effect, the image of the half plane obstacle).

The path of integration in (50) is arbitrary except that we assume the path can cross the real axis only on the segment CK, between $z = -A$ and $z = -a$. To ensure that ψ so defined is independent of the path, we require that

$$f(z) = O(1/z^2) \text{ as } z \longrightarrow \infty \tag{54}$$

f(z) is real and integrable on the slit BCD (55)

To complete the solution, we must now choose f(z) so that ρ, u, v, B_x, B_y as obtained from (50) by (28)-(32) satisfy the regularity conditions, and so that v satisfies the boundary conditions.

8. Solution of the Problem.

In this section we recapitulate the formulation of the diffraction problem, and then propose and verify the solution. We state the conditions of the problem in a stronger form than heretofore. We shall assume that the angle of incidence θ (see Fig. 2) is between 0 and 90° and that $A > a$, though it is a simple matter to give the solution in other cases.

Suppose the incident wave is a fast wave. Then the problem is to find quantities ρ, u, v, B_x, B_y depending only on $(x/t, y/t)$ which satisfy the following conditions:

(1) Except at the curves ABCD, cEaF, GHI, and the line segments OG, OK, KT_1, KL (see Figure 3) the quantities are differentiable and satisfy Equations (22)-(27). Here K is the point

$$x/t = V/\cos\theta = z_o$$

on the half plane obstacle and the front of the incident wave.

(2) Outside the region bounded by the two characteristic lines KT_1, KL and the arc T_1BCDL of the characteristic cone, the quantities are all zero.

(3) On the characteristic cone the quantities are continuous, except at T_1, L, G, at which point the quantities are bounded

(4) Across the line OG, KT, KL the discontinuities in the various quantities satisfy the conditions (34)-(39). These conditions are equivalent to the following:

(a) Across OG, $a^2[\rho] + A^2[B_x] = 0$, $[v]= 0$, $[B_y)= 0$

(b) Across KT_1,

$$[\rho]: [u]: [v]: [B_x]: [B_y] =$$

$$= \frac{1}{a^2} [1-A^2(\lambda^2 + u^2)]: \lambda[1-A^2(\lambda^2 + \mu^2)]: \mu: - \mu^2: \lambda\mu$$

where

$$\lambda = \frac{1}{z_o}, \quad \mu = (\frac{c}{aA}) \frac{1}{z_o}\sqrt{\frac{(z_o^2-A^2)(z_o^2-a^2)}{(z_o^2-c^2)}}, \quad z_o > A .$$

(Using the positive square root)

(c) Across KL, a like condition holds, with

$$\lambda = \frac{1}{z_o}, \quad \mu = - (\frac{c}{aA}) \frac{1}{z_o}\sqrt{\frac{(z_o^2-A^2)(z_o^2-a^2)}{(z_o^2-c^2)}}$$

(5) At the half plane, on OK, the quantities tend to limits for $y/t \to +0$ or $y/t \to -0$, and

for $0 < x/t < z_o$

we have

$$v = -1$$

At one point 0 there is a singularity. Near 0, ρ, u, v, are bounded, but

$$B_x = O\left(\frac{\sqrt{t}}{\sqrt{x^2+y^2}}\right) \text{ and } B_y = O\left(\frac{\sqrt{t}}{\sqrt{x^2+y^2}}\right) .$$

(6) In each of the regions AKT_1, AKL, aEc, aFc, GHI, the quantities ρ, u, v, B_x, B_y are constants. (However, the constants are different for the different regions.)

With these assumptions one has boundary conditions on the inner and outer boundaries of R_1, and corresponding boundary conditions in the z-plane, which we now state. (See Fig. 5)

(7) on both sides of BCD,

$$\rho = u = v = B_x = B_y = 0$$

(8) on both sides of KG,

ρ = constant, u = constant, v = constant,

B_x = constant, B_y = constant

(9) Across OG, v and B_y are continuous and the jumps in ρ and B_x are related by

$$a^2[\rho] + A^2[B_x] = 0$$

(10) On $\eta = \pm 0$, $0 < \xi < z_o$, we have

$$v = 1$$

On $\eta = \pm 0$, $\xi > z_o$ we have

$$\rho = 0,\ u = 0,\ v = 0,\ B_x = 0,\ B_y = 0$$

(11) On $\eta = +0$, $c < \xi < a$, the quantities ρ, u, B_x, B_y, v are constant.

On $\eta = -0$, $c < \xi < a$, the quantities ρ, u, B_x, B_y, v are constant (but not necessarily equal to their values for $\eta = +0$).

(12) The quantities ρ, u, v, B_x, B_y are continuous except at $z = -c$, $z = 0$, $z = z_o$. At $z = -c$ and $z = z_o$ these quantities are bounded. At $z = 0$, the quantities ρ, u, v are bounded, but

$$B_x = 0\left(\frac{1}{|z|^{1/2}}\right),\quad B_y = 0\left(\frac{1}{|z|^{1/2}}\right)$$

From (51) and the equations (28) to (32) expressing ρ, u, v, B_x, B_y in terms of third derivatives of $\psi_\circ$ and using (52) and (46) we find that a possible solution for ρ, u, v, B_x, B_y is represented by

$$
\begin{aligned}
\rho &= \operatorname{Im}\int_\infty^z P(s)ds &&; \quad \frac{\partial\rho}{\partial\xi} = \operatorname{Im} P(z) \\
u &= \operatorname{Im}\int_\infty^z U(s)ds &&; \quad \frac{\partial u}{\partial\xi} = \operatorname{Im} U(z) \\
v &= \operatorname{Im}\int_\infty^z U(s)ds &&; \quad \frac{\partial v}{\partial\xi} = \operatorname{Im} V(z) \\
B_x &= \operatorname{Im}\int_\infty^z B_1(s)ds &&; \quad \frac{\partial B_x}{\partial\xi} = \operatorname{Im} B_1(z) \\
B_y &= \operatorname{Im}\int_\infty^z B_z(s)ds &&; \quad \frac{\partial B_y}{\partial\xi} = \operatorname{Im} B_2(z)
\end{aligned}
\tag{56}
$$

where

$$
P(z) = \left(\frac{c}{aA}\right) \frac{z\sqrt{z^2-A^2}}{\sqrt{(z^2-a^2)(z^2-c^2)}} V(z)
$$

$$U(z) = \left(\frac{ac}{A}\right) \frac{\sqrt{z^2-A^2}}{\sqrt{(z^2-a^2)(z^2-c^2)}} \; V(z)$$

$$B_1(z) = \left(\frac{c}{aA}\right) \frac{1}{z} \frac{\sqrt{(z^2-a^2)(z^2-A^2)}}{(z^2-c^2)} \; V(z) \qquad (57)$$

$$B_2(z) = -\frac{1}{z} V(z) \; .$$

We must now select a function V(z) such that $\frac{\partial \rho}{\partial \xi}$, etc. have the right behavior on the real axis $\eta = 0$, such that

$$P(z), \text{ etc.} = 0(1/z^2) \text{ as } z \longrightarrow \infty$$

and such that v jumps from 0 to 1 as ξ crosses the point

$$\xi = z = z_o \; .$$

The last requirement means that V(z) has a simple pole at $z = z_o$.

It can easily be verified that all requirements are met by the choice

$$V(z) = Q \frac{(z+a)^{1/2}}{z^{1/2}(z+c)^{1/2}(z+A)^{1/2}(z-z_o)} \tag{58}$$

where

$$Q = - \frac{1}{\pi} \frac{z_o^{1/2}(z_o+c)^{1/2}(z_o+A)^{1/2}}{(z_o+a)^{1/2}} \tag{59}$$

Then we have

$$P(z) = (\frac{c}{aA}) Q \frac{z^{1/2}(z-A)^{1/2}}{(z-a)^{1/2}(z-c)^{1/2}(z+c)(z-z_o)} \tag{60}$$

$$U(z) = (\frac{ac}{A}) Q \frac{(z-A)^{1/2}}{z^{1/2}(z-a)^{1/2}(z-c)^{1/2}(z+c)(z-z_o)} \tag{61}$$

$$B_1(z) = (\frac{c}{aA}) Q \frac{(z+a)(z-a)^{1/2}(z-A)^{1/2}}{z^{3/2}(z-c)^{1/2}(z+c)(z-z_o)} \tag{62}$$

$$B_2(z) = -Q \frac{(z+a)^{1/2}}{z^{3/2}(z+c)^{1/2}(z+A)^{1/2}(z-z_o)} \, . \tag{63}$$

Condition (9) on the jumps in ρ, B_x across OG is easily verified. Also it is easily verified that the jumps in ρ, u, v, B_x, B_y at $z = z_o$ are in accordance with conditions (4b) and (3).

Thus ρ, u, v, B_x, B_y are obtained (by integrating these formulas, in accordance with (56), as functions of z. The functions involved are elliptic integrals of the third kind. To obtain ρ, u, v, B_x, B_y as functions of (x/t), (y/t) we must obtain z from (52), (53). In general this involves solving a biquadratic equation. However, for $y/t = 0$, which is an interesting special case, matters are simpler, for here, as has been stated,

$$z = x/t \ .$$

If the incident wave is slow, the solution is given by the same formulas (58)-(63) except that for a slow incident wave we have

$$c < z_o < a \ .$$

9. Discussion of the Results.

We now mention some features of the pattern of the reflected and diffracted waves that have been revealed by our analysis.

If the incident wave is fast, we have the situation illustrated in Fig. 3.

Ahead of MT_1AK the fluid is at rest.

Behind MT_1CDLKN we have the incident wave as yet undisturbed by the half plane.

In KAL the flow is a constant state (that is, ρ, u, v, B_x, B_y are constant throughout KAL). The flow in KAL is just like the flow for a fast wave reflected from a full plane.

In GHI, and also in CaE, and in caF the flow is a constant state.

At OG there is a contact discontinuity. Across OG the parallel component of velocity u is discontinuous, and also the density ρ and the component B_x of the magnetic field. Since ρ is discontinuous, so is the pressure. Since B_x is discontinuous, a sheet current flows in the z direction in OG. The pressure jump is balanced by the force exerted by $\vec{B}_o$ on this sheet current, and so we have equilibrium. This is the content of condition (4a).

At the edge O of the half plane, ρ, u, v, are continuous (on the upper side and on the lower side), but B_x, B_y become infinite like the reciprocal square root of the distance from the edge.

If the incident wave is slow, we have the situation illustrated by Fig. 4.

Ahead of MVBALN the fluid is at rest.

Behind MVCDLN the incident wave is as yet undisturbed by the half plane.

The quantities ρ, u, v, B_x, B_y suffer jumps across VK, T_3L — which lines are the front of the incident wave propagating through the diffracted wave which in this case outruns the incident wave.

Across KU the flow has a jump discontinuity. The line KU is the front of the reflected wave, propagating through the diffracted wave.

The flow is continuous across KT_3, which is the shadow.

In the following regions the flow is a constant state: GHI, cKW, WKU, EUKa, $cKaPT_3$, T_3PF.

A remarkable feature of the results is the occurrence of regions of constant state inside the cusped curves GHI, cKW. This is another manifestation of a phenomenon discovered by Weitzner(4) for a different problem. Weitzner found the fundamental solution of the system (22)-(27). (That is, the solution of (22)-(27) modified by replacing the zeros on the right-hand sides by Dirac delta functions.) Weitzner found that the interiors of the cusped curves are lacunae; that is, the fundamental solution is identically zero in these regions.

Appendix. Mapping of the (x/t), (y/t)-Plane into the z-Plane.

In this Appendix we wish to show that the equations

$$\lambda(z)(\frac{x}{t}) + \mu(z)(\frac{y}{t}) = 1 \tag{64}$$

$$\lambda = \frac{1}{z}, \ \mu = (\frac{c}{aA}) \ \frac{1}{z}\sqrt{\frac{(z^2-a^2)(z^2-A^2)}{(z^2-c^2)}} \tag{65}$$

define a differentiable one-to-one mapping of the z-plane slit as in Fig. 5 into a region R_1 of the (x/t), (y/t) plane (shown in Fig. 1), with the following properties:

(1) A point T on the boundary of R_1 is mapped into the point U where the tangent TU cuts the (x/t) axis.

(2) Points in R_1 for which $y/t = 0$ are invariant.

(3) Near 0 the mapping approximates the identity mapping in that

$$z = (\frac{x}{t} + i \ \frac{y}{t})[1 + 0(|z|^4)] \ . \tag{66}$$

(4) The bounding curves of R_1 determine characteristic surfaces.

To show that the mapping is one-to-one we show first that for a complex value of z, the point (x/t, y/t) is determined uniquely, and is a differentiable function of ξ, η where ξ, η are the real and imaginary parts of z. Let λ_1, μ_1 be the real parts of λ, μ and λ_2, μ_2 the imaginary parts. The real and imaginary parts of (64) yield two linear equations, and the solution is

$$\frac{x}{t} = \frac{\mu_2}{\lambda_1\mu_2 - \lambda_2\mu_1} \tag{66}$$

$$\frac{y}{t} = \frac{-\lambda_2}{\lambda_1\mu_2 - \lambda_2\mu_1} \;.$$

We must verify that the denominator is not zero. The denominator is

$$\lambda_1\mu_2 - \lambda_2\mu_1 = -(\mu_1^2 + \mu_2^2) \ \mathrm{Im}\ \left(\frac{\lambda_1 + i\lambda_2}{\mu_1 + i\mu_2}\right)$$

which is

$$-\left(\frac{aA}{c}\right)(\mu_1^2 + \mu_2^2)\ \mathrm{Im}\left\{\sqrt{\frac{(z^2-c^2)}{(z^2-a^2)(z^2-A^2)}}\right\} \;.$$

The imaginary part of a function analytic in some domain takes on its maximum at the boundary of the domain. Take the domain to be the slit z-plane of Fig. 5 with small regions excised near $z = \pm a$, $z = \pm A$. Near these points it is easily shown that

$$\mathrm{Im}\left\{\sqrt{\frac{z^2-c^2}{(z^2-a^2)(z^2-A^2)}}\right\} < 0$$

and on the slits we have

$$\mathrm{Im}\left\{\sqrt{\frac{z^2-c^2}{(z^2-a^2)(z^2-A^2)}}\right\} = 0 .$$

It follows that in the interior of the entire slit z-plane we have

$$\mathrm{Im}\left\{\sqrt{\frac{z^2-c^2}{(z^2-a^2)(z^2-A^2)}}\right\} < 0 \; ; \text{ so } \lambda_1\mu_2 - \lambda_2\mu_1 > 0. \qquad (67)$$

Hence a differentiable mapping of the z-plane into the (x/t), (y/t) plane is defined by (64), (65).

By (66), (67), we see that if z is in the upper half plane, so that λ_2 is negative, then

$$y/t > 0$$

and for z in the lower half plane,

$$y/t < 0$$

Thus Im(z) and (y/t) have the same sign.

Now to show that the mapping is one-to-one, we show that for given (x/t), (y/t) there is not more than one value of z in the interior of the slit z-plane such that (64), (65) hold.

The possible such values of z are among the four roots of the biquadratic equation

$$[\mu(z)(\tfrac{y}{t})]^2 = [1-\lambda(z)(\tfrac{x}{t})]^2 .$$

This equation always has a real root z with $-a \leq z \leq -c$, and a real root such that $c \leq z \leq a$. We see this by noting that the family of lines

$$\frac{1}{z}(\frac{x}{t}) \pm \frac{c}{aA}\,\frac{1}{z}\sqrt{\frac{(z^2-a^2)(z^2-A^2)}{(z^2-c^2)}}\,(\frac{y}{t}) = 1, \quad -a \leq z \leq -c$$

covers the whole (x/t), (y/t) plane, since the intercept of such a line with the (y/t) axis can have all values from $-\infty$ to $+\infty$. A similar observation shows the existence of a root with $c \leq z \leq a$, hence the biquadratic equation can have only two roots with

$$\operatorname{Im}(z) \neq 0.$$

These roots are conjugate complex numbers, and for only one of them can $\operatorname{Im}(z)$ have the same sign as (y/t). This proves the mapping is one-to-one, except for the cases $-A < z < -a$, $-c < z < c$, $a < z < A$. But for such a value of z, the quantity μ is pure imaginary, and λ is real; that is,

$$\lambda_2 = 0, \mu_1 = 0$$

and from (66) we get

$$\eta = 0, \xi = z = (x/t)$$

which of course is an unique value of z.

By the way, we have moved property (2) above.

Now let us consider the mapping of the boundary. We consider

$$z = \xi + i\eta$$

with

$$\xi < -A, \text{ or } -a < \xi < -c, \text{ or } c < \rho < a, \text{ or } A < \xi,$$

and assume that η is small. For $\eta = 0$ the point (ξ, η) is on some part of the boundary of the region in the z-plane. Also, for $\eta = 0$, λ and μ are real. Expanding in powers of η we see that for small η the real and imaginary parts of (64) are

$$\begin{aligned} &\lambda(\xi)\left(\frac{x}{t}\right) + \mu(\xi)(y/t) = 1 \\ &\eta\left\{\frac{\partial}{\partial\xi}\lambda(\xi)\left(\frac{x}{t}\right) + \frac{\partial}{\partial\xi}\mu(\xi)\left(\frac{y}{t}\right)\right\} = 0 \, . \end{aligned} \tag{67}$$

Here the sign of μ will depend on whether η is positive or negative — that is on from which side the point (ξ, η) tends to the slit as $|\eta| \longrightarrow 0$. From (67) we obtain the following equations, which determine the image, in the (x/t), (y/t) plane, of the bounding slits in the z-plane:

$$\begin{aligned} &\lambda(\xi)(x/t) + \mu(\xi)(y/t) = 1 \\ &\lambda'(\xi)(x/t) + \mu'(\xi)(y/t) = 0 \end{aligned} \tag{68}$$

The determinant of these equations is not zero; for this determinant is

$$\lambda\mu' - \mu\lambda' = \lambda^2 \frac{\partial}{\partial\xi}\left(\frac{\mu}{\lambda}\right) = \frac{\lambda^3}{2\mu}\frac{\partial}{\partial\xi}\left[\left(\frac{\mu}{\lambda}\right)^2\right]$$

and direct computation shows that

$$\frac{\partial}{\partial\xi}\left[(\mu/\lambda)^2\right] = 0$$

only for

$$\xi = 0 \ , \ \xi = \pm\sqrt{\frac{2a^2A^2}{a^2+A^2}}$$

and, since $A > a$ we have

$$A = \sqrt{\frac{2a^2A^2}{a^2+a^2}} > \sqrt{\frac{2a^2A^2}{a^2+A^2}} > \sqrt{\frac{2a^2A^2}{A^2+A^2}} = a \ .$$

Thus at no point ξ under consideration, (so that $(\xi, \pm 0)$ is on one of the bounding slits), does the determinant vanish.

Now note that Equation (68) says that the point (x,y,t) is on the envelope of the family of characteristic planes given by

$$\lambda(\xi)\,x + \mu(\xi)y = t.$$

(These planes are characteristic, for we have, identically,

$$F(\lambda,\mu) = 1-(a^2+A^2)(\lambda^2+\mu^2)+a^2A^2\lambda^2(\lambda^2+\mu^2) = 0.$$

Thus we have proved assertion (4), that the boundary curves in the plane of $(x/t,\ y/t)$ represent characteristic surfaces.

Assertion (1) follows also; for the line

$$\lambda(\xi)(\overline{x/t}) + \mu(\xi)(\overline{y/t}) = 1$$

is tangent to the envelope at

$$\left(\overline{\frac{x}{t}}\right) = \frac{x}{t}\ ,\quad \left(\overline{\frac{y}{t}}\right) = \frac{y}{t}$$

and the intercept of this line on the (x/t) axis is at

$$\left(\frac{\bar{x}}{t}\right) = \frac{1}{\lambda(\xi)} = \xi = z.$$

To move assertion (3), we expand $\mu(z)$ in powers of z. We have

$$z\mu = i\sqrt{\frac{(1-z^2/a^2)(1-z^2/A^2)}{(1-z^2/c^2)}}$$

where in this formula we define the square root so that it is +1 at $z = 0$. Then from

$$\lambda(x/t) + \mu(y/t) = 1$$

we obtain

$$\left(\frac{x}{t}\right) + i\left(\frac{y}{t}\right)\sqrt{\frac{(1-z^2/a^2)(1-z^2/A^2)}{(1-z^2/c^2)}} = z.$$

Expanding in powers of z, we have

$$\left(\frac{x}{t}\right) + i\left(\frac{y}{t}\right)[1+O(|z|^4)] = z$$

whence assertion (3) follows.

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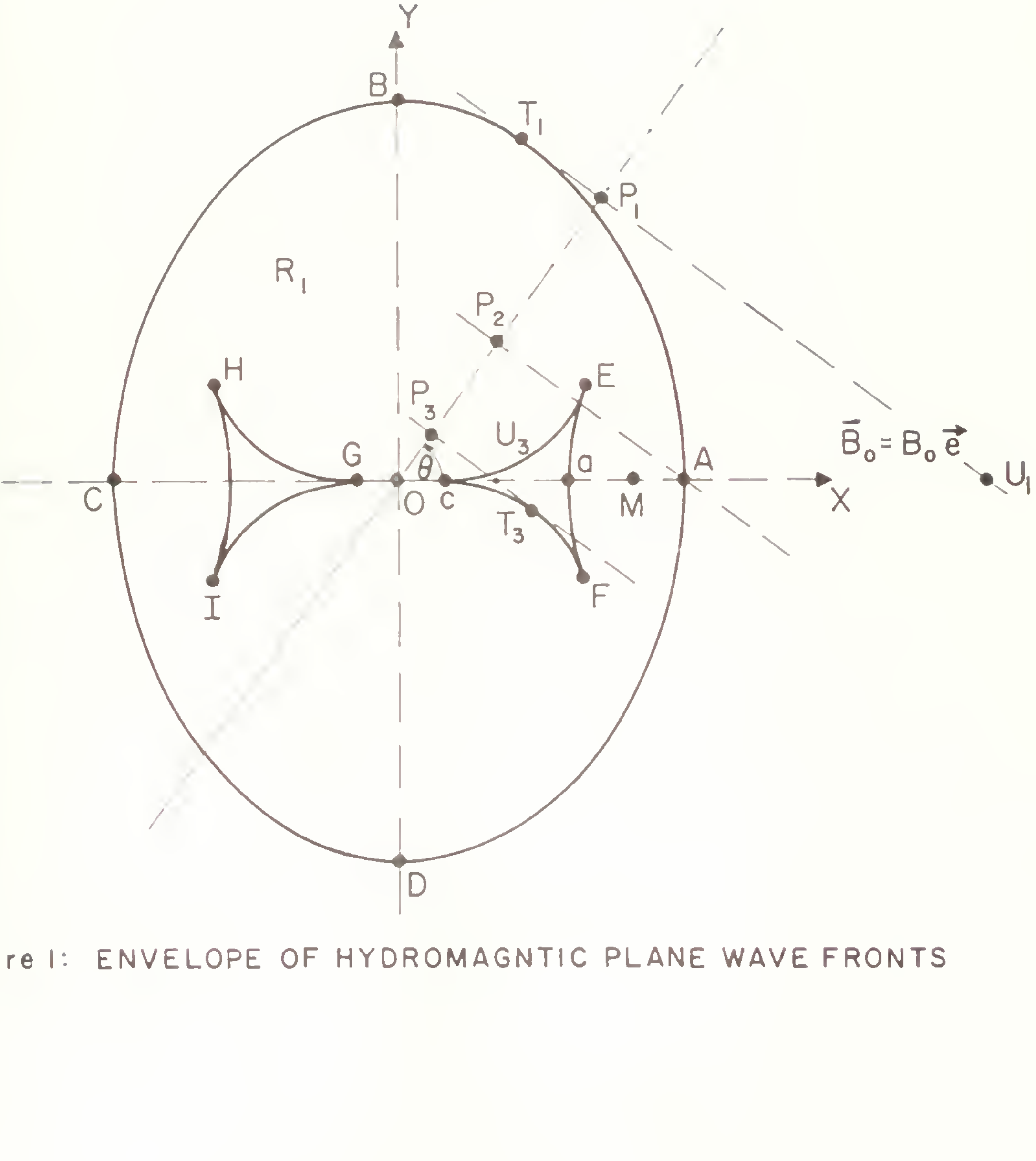

Figure I: ENVELOPE OF HYDROMAGNTIC PLANE WAVE FRONTS

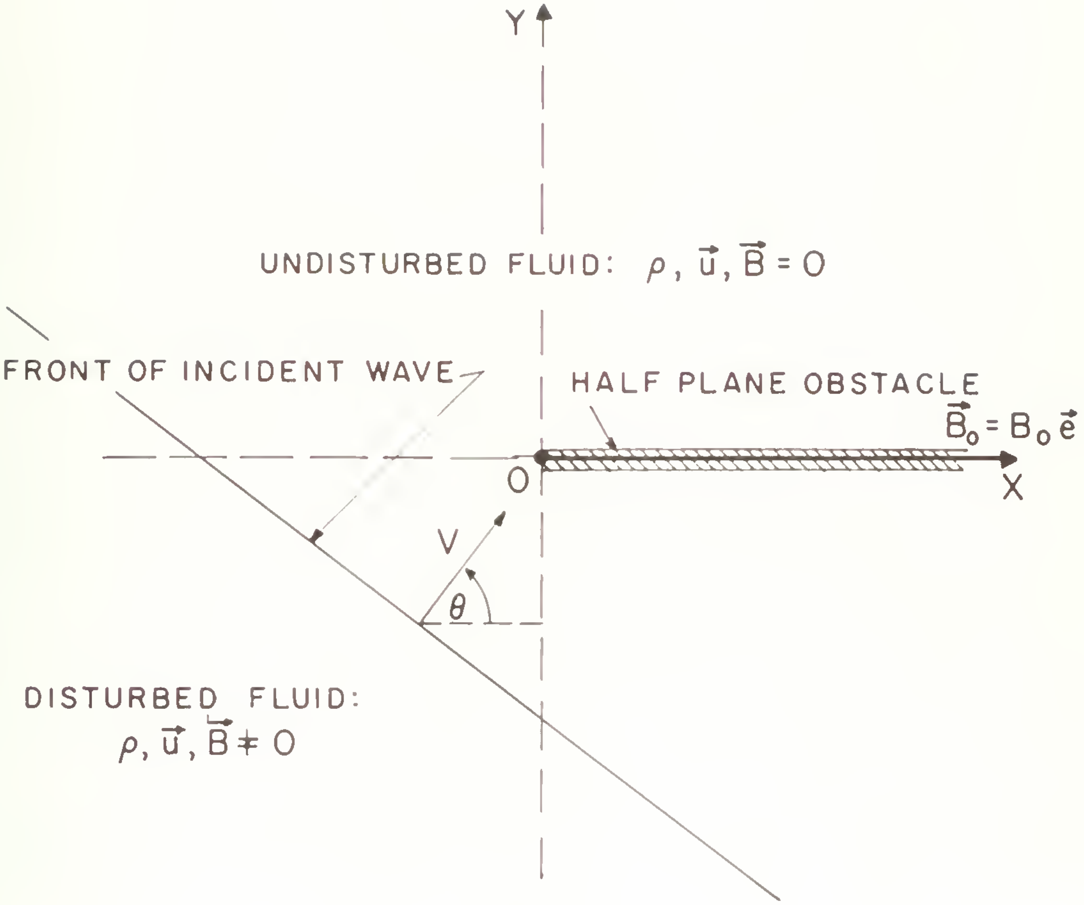
Y
UNDISTURBED FLUID: $\rho, \vec{u}, \vec{B} = 0$
FRONT OF INCIDENT WAVE
HALF PLANE OBSTACLE
$\vec{B}_0 = B_0 \vec{e}$
O
X
V
θ
DISTURBED FLUID:
$\rho, \vec{u}, \vec{B} \neq 0$

Figure 2: STATE OF FLUID BEFORE INCIDENT WAVE STRIKES HALF-PLANE

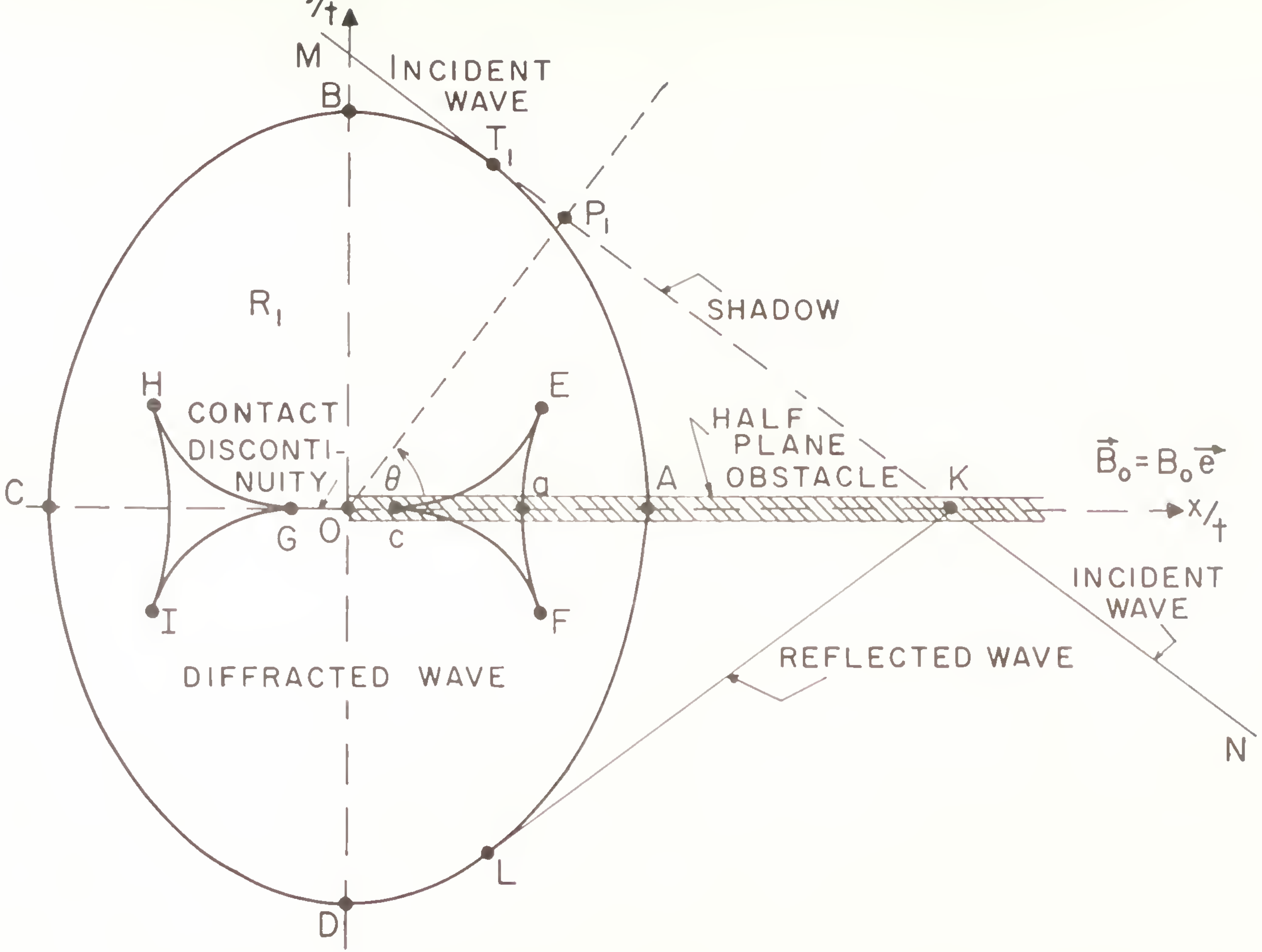

Figure 3: DIFFRACTION OF FAST SHOCK BY HALF PLANE

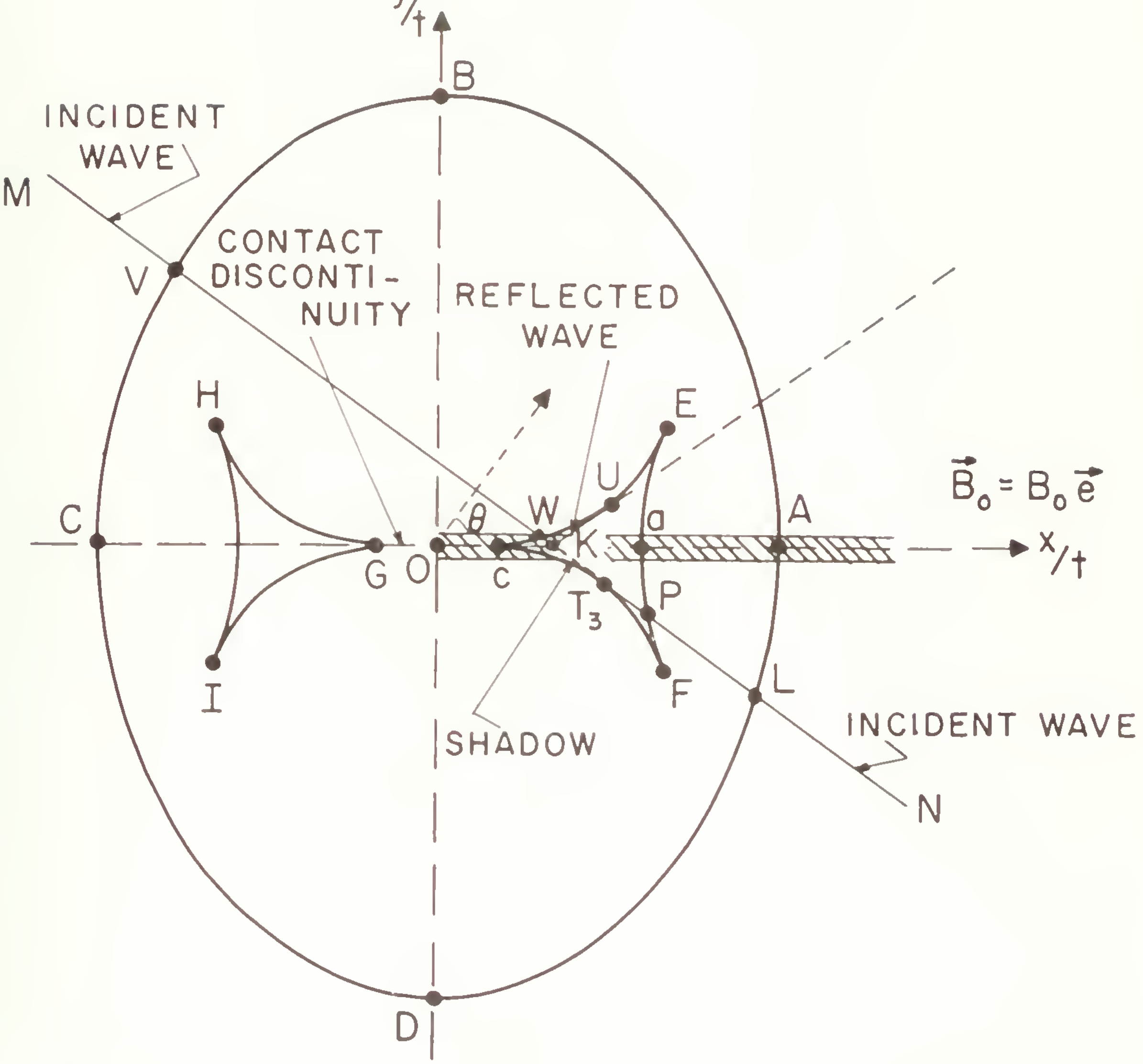

Figure 4: DIFFRACTION OF SLOW SHOCK BY HALF PLANE

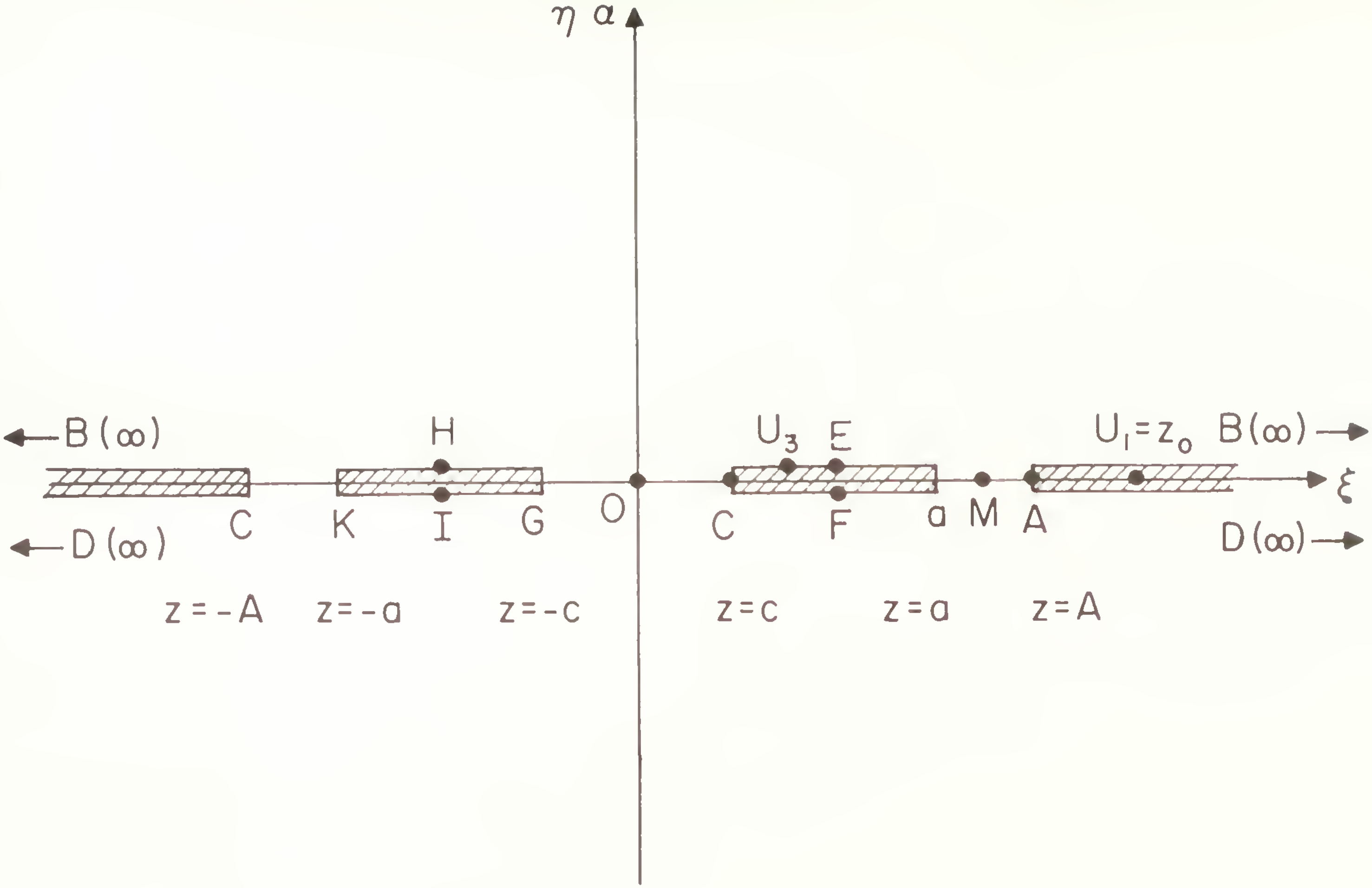

Figure 5: THE z-PLANE. HERE $z = \xi + i\eta$.

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N. Y. U. Institute of
Mathematical Sciences
25 Waverly Place
New York 3, N. Y.